Criteria for Exact Solubility of Relativistic Field Theories by Scattering Transform

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Abstract

Scattering transform is a well known powerful tool for quantisation of field theories in (1+1) dimensions. Conventionally only those models whose classical counterparts admit a Lax pair (origin of which is always mysterious) have been quantised in this way. In relativistic quantum field theories we show that the scattering transforms can be constructed ab initio from its invariance under Lorentz transformation (both proper and improper), irreducible transformation nature of scalar and Dirac fields, the existence of a momentum scale associated with asymptotic nature of the scattering transform and the closure of short distance operator product algebra. For single fields it turns out that theories quantisable by scattering transforms are restricted to sine-Gordon type for spin-0 and Massive Thirring type for spin-$\frac{1}{2}$ if the target space of the scattering transform matrix is assumed to be parity invariant. There are interesting unexplored extensions if the target space is given chirality.

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Just as free field theories can be exactly solved by showing that the Hamiltonian, expressed in terms of the components of the Fourier transforms of the fields and their canonical conjugates, is cleanly diagonalisable – quite a few nonlinear field theories are also known to be integrable in an equivalent sense that a certain scattering transform give rise to a set of new variables in terms of which the Hamiltonian is exactly diagonalisable[1]. These transforms are constructed from a constant time path ordered integral (Wilson line [3]) of a local Lie algebra valued object called the Lax operator (or the scattering operator since, for $x \to \pm\infty$, it takes a value of a constant momentum k and hence the integral itself has asymptotic plain wave structure). Expressed mathematically, the scattering matrix $\mathcal{T}(x, y; k)$ is given by

$$\mathcal{T}(x, y; k) = \mathcal{P} \exp\left[i \int_y^x L(\xi, k) d\xi\right] \tag{1}$$

$$L(x, k) = \sum_i^N t^i(k) j_i(x) \to T_d k \quad \text{as } x \to \pm\infty \tag{2}$$

Here $t^i(k)$ are the generators of a graded algebra in the form

$$t^i(k) = \sum_a f_a^i(k) t^a \tag{3}$$

where t^a's are generators of a simple Lie algebra (normally taken in its fundamental representation for simplicity in algebraic manipulation) and $f_a^i(k)$ are appropriately chosen functions of the momentum k. T_d is in the Cartan sub-algebra (hence taken diagonal). The dynamical information are all in the N number of local fields j_i's characterised by their non-trivial equal time commutation algebra

It is known [1, 2] that for some appropriate choices of $L(x, k)$ the scattering marix would satisfy the following braid algebra

$$\mathcal{R}(k, q)[\mathcal{T}(x, y; k) \otimes I][I \otimes \mathcal{T}(x, y; q)] = [I \otimes \mathcal{T}(x, y; q)][\mathcal{T}(x, y; k) \otimes I]\mathcal{R}(k, q) \tag{4}$$

Here $\mathcal{R}$, called the braiding matrix that acts on the product representation space of the two scattering transforms, depends on the spectral variables k and q only. This automatically lead to the involution relation

$$[\mathrm{Trace}(\mathcal{T}(k)), \mathrm{Trace}(\mathcal{T}(q))] = 0 \tag{5}$$

where $\mathcal{T}(k)$'s are the actual scattering matrics in the sense that one has taken $x \to \infty$ and $y \to -\infty$ with the asymptotic oscillating plane wave components factored out. This involution algebra leads to the existence of infinite number of mutually commuting objects which are the coefficients of different powers of k after Taylor expanding $\mathrm{Trace}(\mathcal{T}(k))$ in (inverse) powers of k. One of these coefficients when evaluated as an integral in terms of the basic field variables (usually the coefficient of k^{-2} for non-relativistic theories) would look like a Hamiltonian of a nonlinear theory. This means that such a Hamiltonian system has infinite number of constants of motion, all mutually commuting. In that sense these quantum field theories are integrable. The explicit form of $\mathcal{R}$ also enables one to study the evolution pattern for the non-diagonal operators of the scattering matrix $\mathcal{T}$ by studying their commutation algebra with the diagonal elements. These non-diagonal operators are

the counterparts of the Fourier transforms and hence are appropriately called the *scattering transforms*. All the eigenvalues any of the conserved quantities including the Hamiltonian can subsequently be found from the commutation algebra between the constants of motion and the scattering transforms [1].

Exactly what forms of $L(x,k)$ would finally lead to the involution relation like Eq.(5) has always remained a big mystery. Traditionally one picks up those classical nonlinear models which admit Lax pairs [4] and choose the one corresponding to the space variation as the L operator of Eq.(2). What we establish in this paper is that in the relativistic theories in 1+1 dimensions the choices for such L cannot be too many. For this one need not start with any such classically integrable system but use the properties of local quantum fields and arrive at a consistent form of L that would satisfy Eq.(4 and hence the integrability (the classical limit of Eq.(5)). We show now how the following conditions put such severe restriction on the form of L.

Relativistic covariance:

It is clear from Eq.(1) that the scattering matrix is a connection term - something like a Wilson line in a non-Abelian gauge theory and for the truly infinite limit it has to be a Lorentz invariant quantity. Moreover, to clearly distinguish between momentum and Hamiltonian from their Lorentz covariance we have to make $\mathcal{T}(x,y;k_1)$ parity and time reversal invariant too. *This means $L(x,k_1)$ has to be chosen in such a way that it would transform like the space component of a true vector.* Here we are assuming that the target space (the representation space on which $\mathcal{T}$ acts) is parity invariant. In the later part of the paper we will relax this condition to explore the possibilities of existence of other soluble models.

The next thing that one has to remember that each fundamental local field must transform irreducibly under Lorentz transformation (LT) that itself acts irreducibly on the light cone variables,

$$x^0 \pm x^1 = x_\pm \rightarrow e^{\pm\theta} x_\pm \tag{6}$$

where θ is the boost parameter. Scalar fields are invariant under LT whereas Dirac fields transform as

$$\psi_{1,2} \rightarrow e^{\pm\theta/2} \psi_{1,2} \tag{7}$$

Finally one has to think about the LT of the spectral parameter that enters into the picture from the asymptotic behaviour where the fields are supposed to vanish in the matrix element sense. Instead of taking k_1, we will take the irreducible objects $k_0 \pm k_1 = k_\pm$ with $k_+ k_-$ a Lorentz invariant constant and that we identify with a given mass scale of the theory. Hence the appropriate dimensionless spectral parameter is taken to be λ such that

$$k_+ = m\lambda \quad \text{and} \quad k_- = \frac{m}{\lambda} \tag{8}$$

One important consequence of Lorentz covariance is that the grading functions $f_a^i(\lambda)$ occurring in L must be a monomial in λ with the power determined by the nature of LT of the Lorentz irreducible component $j_i(x)$ of Eq.(2). For example, if $j_i(x)$ has 'spin' n_i i.e., it transforms irreducibly under LT as

$$j_i \rightarrow e^{n_i\theta} j_i \tag{9}$$

the associated grading function must transform irreducibly as

$$f_a^i(\lambda) \to e^{(-n_i \pm 1)\theta} f_a^i(\Lambda^{-1}\lambda)$$

implying

$$f_a^i(\lambda) = c_a^i \lambda^{-n_i \pm 1} \tag{10}$$

Since this is multiplied by the generators t^a of a Lie algebra, the constants c_a^i's just cause a linear combination of the old basis t^a to a new basis T^i and in Lax operator these are just graded by a single power of the spectral parameter - a considerable reduction from the original general form suggested in Eq.(3)! The changed basis will be complete if the number of fields (N) is exactly equal to the dimension of the Lie algebra. However, this matching is not necessary. We may have a under-complete basis (as in Toda field theory) or an over-complete basis (as in massive Thirring model to be discussed in this paper itself).

Parity Invariance:

If one insists on parity invariance, i.e. if one demands that under parity $L(x,\lambda) \to -L(x,\lambda)$, one can remove the ± 1 ambiguity in the grading. L, like the space component of momentum, does not transform irreducibly under LT. It must have one part that will transform like λ and another that transforms like $1/\lambda$. In other words

$$L(x,\lambda) = L_+(x,\lambda) - L_-(x,\lambda) \tag{11}$$

with $L_\pm(x,\lambda) \to e^{\pm\theta} L_\pm(\Lambda^{-1}x,\lambda)$ and $L_+ \longleftrightarrow L_-$ under parity. In such a situation we can grade separately the generators of $L_\pm$ by the functions $f_{a,\pm}^i(\lambda)$ and consequently, using the LT restriction –

$$\begin{array}{rcl} f_{a,+}^i(\lambda) &=& c_a^i \lambda^{-n_i^+ + 1} \\ f_{a,-}^i(\lambda) &=& c_a^i \lambda^{-n_i^- - 1} \end{array} \tag{12}$$

where $n_i^\pm$ are the 'spin' indices of the parity conjugate fields $j_i^\pm$, $j_i^+ \overset{\text{parity}}{\longleftrightarrow} j_i^-$ and

$$L_\pm(x,\lambda) = f_{a,\pm}^i(\lambda) t^a j_i^\pm(x) \tag{13}$$

Causality:

One main reason for taking a constant time path for the scattering transform is to ensure that all local fields are causally separated. This makes the operator $\mathcal{T}(x,y;\lambda)$ well defined in quantum theory. The quantum dynamics enter the picture when we take exterior product of one $\mathcal{T}(x,y;\lambda)$ with another to check the existence of braiding relation. This is where we will come across products of fields in the equal time short distance limit. To have a non-trivial braiding (i.e. getting a $\mathcal{R}(\lambda,\mu)$ different from identity matrix and depending on the spectral parameters), we must encounter products that would have singularities in the equal time short distance limit. This can happen only when $L(x,\lambda)$ contains canonically conjugate fields, their polynomials and even entire functions. One should note here that we are translating the equal time commutator, generally used in non-relativistic integrable systems, into a more symmetric looking operator product expansion with singularities.

Closure of Operator Product Algebra:

The choice of fields to be included in $L(x,\lambda)$ must satisfy another important condition, namely the *closure* under operator product expansion (OPE)[5]

$$T(j_i(x)j_j(y)) = \sum_{n\geq 1} \frac{{}^{(n)}O_{ij}(x)}{(x-y-i0^+)^n} + \quad \text{regular terms} \tag{14}$$

The closure implies that all the local operators ${}^{(n)}O_{ij}(x)$ (including identity) associated with the pole singularities of any pair of fields occurring in $L(x,\lambda)$ must themselves occur in $L(x,\lambda)$. To prove this point, consider the products of two infinitesimal Wilson lines $\mathcal{T}_\epsilon(x;\lambda)$ and $\mathcal{T}_\epsilon(x;\mu)$, where

$$\mathcal{T}_\epsilon(x;\lambda) \equiv \mathcal{T}(x+\epsilon/2, x-\epsilon/2; k) = 1 + i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} t^i(\lambda) j_i(\xi) d\xi + \mathcal{O}(\epsilon^2) \tag{15}$$

If there were no singularities in the OPE, this product, to order ϵ, will have exactly the same look as the classical product, namely

$$\begin{aligned}
[\mathcal{T}_\epsilon(x;\lambda)\otimes 1][1\otimes \mathcal{T}_\epsilon(x;\mu)] &= 1\otimes 1 + i[(t^i(\lambda)\otimes 1) + (1\otimes t^i(\mu))]\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} j_i(\xi)d\xi \\
&= [1\otimes \mathcal{T}_\epsilon(x;\mu)][\mathcal{T}_\epsilon(x;\lambda)\otimes 1]
\end{aligned}$$

implying trivial braiding ($\mathcal{R}=1$). When there are singularities in the OPE, there will be more terms to order ϵ [6]. For example,

$$(t^i(\lambda)\otimes t^j(\mu)) \int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} j_i(\xi)d\xi \int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} j_j(\eta)d\eta$$

which was classically of order ϵ^2 and hence ignored, will now be of order ϵ if $j_i(\xi)j_i(\eta)$ has a first order pole singularity. If the local field associated with this singularity is ${}^{(1)}O_{ij}$ then, to order ϵ, one will have

$$\begin{aligned}
[\mathcal{T}_\epsilon(x;\lambda)\otimes 1][1\otimes \mathcal{T}_\epsilon(x;\mu)] &= 1\otimes 1 + i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} i[(t^i(\lambda)\otimes 1) + (1\otimes t^i(\mu))]j_i(\xi)d\xi \\
&\quad + c\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} (t^i(\lambda)\otimes t^j(\mu))^{(1)}O_{ij}(\xi)d\xi, \\
\text{whereas} & \\
[1\otimes \mathcal{T}_\epsilon(x;\mu)][\mathcal{T}_\epsilon(x;\lambda)\otimes 1] &= 1\otimes 1 + i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} i[(t^i(\lambda)\otimes 1) + (1\otimes t^i(\mu))]j_i(\xi)d\xi \\
&\quad + c\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} (t^i(\mu)\otimes t^j(\lambda))^{(1)}O_{ij}(\xi)d\xi
\end{aligned} \tag{16}$$

Clearly $\mathcal{R}$ is not proportional to identity any more. However if $\mathcal{R}$ exists it must satisfy a relation

$$\begin{aligned}
&[\mathcal{R}, (t^i(\lambda)\otimes 1)j_i(x) + (1\otimes t^i(\mu))j_i(x)] \\
&-ic\left(\mathcal{R}t^i(\lambda)\otimes t^j(\mu)^{(n)}O_{ij}(x) - t^i(\mu)\otimes t^j(\lambda)^{(1)}O_{ij}(x)\mathcal{R}\right) = 0
\end{aligned}$$

Now if ${}^{(1)}O_{ij}(x)$ are linearly independent of j_i's one will have two sets of algebraic equations for $\mathcal{R}$,

$$\begin{array}{c}[\mathcal{R},(t^i(\lambda)\otimes 1)+(1\otimes t^i(\mu))]=0\\ \mathcal{R}t^i(\lambda)\otimes t^j(\mu)-t^i(\mu)\otimes t^j(\lambda)\mathcal{R}=0\end{array} \tag{17}$$

The second relation is inconsistent. In the classical limit when $\mathcal{R}$ is close to identity, to the lowest order in Plank constant one would get

$$t^i(\lambda)\otimes t^j(\mu)-t^i(\mu)\otimes t^j(\lambda)=0$$

a relation that can only be true when $\lambda=\mu$ since the power of the monomials are necessarily different.

When the leading singularities of OPE correspond to poles higher than first order the calculation of the quantum correction is not as straightforward as has been done in Eq.(16). We will discuss it in the latter part of the article when we will show that by appropriate gauge transformation of the Wilson line, the singularities of the OPE among the relevant operators in the gauge transfomed L can be always reduced to simple poles. Consequently the same argument as discussed above can be invoked to prove that the local fields j_is occurring in $L(x,\lambda)$ must satisfy a closed OPE, namely,

$$T(j_i(x)j_j(y))=\sum_n \frac{{}^{(n)}F_{ij}^k\ j_k(x)}{(x-y-i0^+)^n}+\quad \text{regular terms} \tag{18}$$

Construction of Lax Operators:

Equipped with the necessary requirements of Lorentz covariance, causality, and closure of local fields in OPE, we now proceed to actual construction of L operators with 'spin' 0 (scalar or pseudo-scalar) and 'spin' $\frac{1}{2}$ (Dirac) fields separately. For the sake of simplicity we will not include any flavour index, though such extensions are incorporable with corresponding enlargement of the Lie algebra.

Consequent to the discussion in the last part of the previous section, we will also confine ourselves, for the time being, to the construction of those L for which OPE of the local fields do not have singularities higher than first order poles.

Spin-0^+*:*

For true scalar fields we encounter a no go situation. Recall that L must contain both ϕ and $\dot{\phi}$ so that the equal time limit of the OPE of L's would be nontrivial (i.e. having singularities). $\dot{\phi}$ which is the canonical conjugate of ϕ transforms as the time component of a vector and cannot occur in L in a simple way. One can try a construed way of introducing a term like $k_1(k.\partial\phi)$ or $k_1(\partial\phi)^2/m^2$ but that will violate the closure property. We could have added a term like $\epsilon_{10}\partial^0\phi$ in L but that will destroy the parity invariance.

Spin-0^- *(Pseudo-scalar)*

For a pseudo-scalar field ϕ, all even functions will be parity invariant while all odd functions change sign. $\partial_+\phi$ will have its parity conjugate as $-\partial_-\phi$ and consequently a general choice for $L_\pm$ will be

$$L_\pm(x,\lambda)=(T^1(\pm\partial_\pm\phi)+T^2\lambda^{\pm1}E(\phi)+T^3\lambda^{\pm1}(\pm O(\phi))$$

Using Eq.(11) we thus obtain

$$L(x,\lambda) = \beta\dot{\phi}t^1 + \frac{m}{2}(\lambda + \frac{1}{\lambda})O(\phi)t^2 + \frac{m}{2}(\lambda - \frac{1}{\lambda})E(\phi)t^3 \tag{19}$$

showing the occurrence of canonically conjugate field and hence the possibility of nontrivial braiding. To get the right asymptotic behaviour we have to impose the condition $O(\phi) \to 0$ and $E(\phi) \to 1$ as $x \to \pm\infty$. To know more about the even and odd functions we have to invoke the closure of OPE algebra. The singularity associated with the OPE of $\dot{\phi}$ with $O(\phi)$ would involve the derivative $O'(\phi)$ which is an even function and vice versa. The closure then imposes the condition

$$O'(\phi) \approx E(\phi) \quad \text{and} \quad E'(\phi) \approx O(\phi) \tag{20}$$

This means O must be a sine (or sinh) function and E must be a cosine (or cosh) function. To study it in detail we start with the OPE algebra of real scalar field

$$T(\phi(x)\dot{\phi}(y)) = \frac{1}{2\pi(x - y - i0^+)} + \quad \cdots$$

consistent with the equal time commutation algebra

$$[\phi(x^1, x^0), \dot{\phi}(y^1, x^0)] = i\delta(x^1 - y^1).$$

This tells us that

$$T(O(\phi(x))\dot{\phi}(y)) = \frac{O'(\phi(x))}{2\pi(x - y - i0^+)} + \quad \cdots$$

$$T(E(\phi(x))\dot{\phi}(y)) = \frac{E'(\phi(x))}{2\pi(x - y - i0^+)} + \quad \cdots$$

If we write $O'(\phi(x)) = \beta_1 E(\phi(x))$ and $E'(\phi(x)) = \beta_2 O(\phi(x))$ we get, after appropriate redefinition of the generators, two possible form of L, namely,

$$L(x,\lambda) = \beta\dot{\phi}t^2 + \frac{m}{2}(\lambda + \frac{1}{\lambda})\sin(\beta\phi)t^1 + \frac{m}{2}(\lambda - \frac{1}{\lambda})\cos(\beta\phi)t^3 \tag{21}$$

$$L'(x,\lambda) = \beta\dot{\phi}t^2 + \frac{m}{2}(\lambda + \frac{1}{\lambda})\sinh(\beta\phi)t^1 + \frac{m}{2}(\lambda - \frac{1}{\lambda})\cosh(\beta\phi)t^3 \tag{22}$$

Using Eq.(16) and exploiting the linear independence of the local fields, we get the master equations that the braiding matrices $\mathcal{R}$ for the scattering transforms of these two cases must satisfy if they exist.

$$\begin{aligned}
&[\mathcal{R}, (t^2 \otimes 1) + (1 \otimes t^2)] = 0 \\
&[\mathcal{R}, (\lambda - \frac{1}{\lambda})(t^3 \otimes 1) + (\mu - \frac{1}{\mu})(1 \otimes t^3)] + \frac{-\beta^2}{2}\left\{\mathcal{R}, (\lambda + \frac{1}{\lambda})(t^1 \otimes t^2) - (\mu + \frac{1}{\mu})(t^2 \otimes t^1)\right\} = 0 \\
&[\mathcal{R}, (\lambda + \frac{1}{\lambda})(t^1 \otimes 1) + (\mu + \frac{1}{\mu})(1 \otimes t^1)] + \frac{\pm\beta^2}{2}\left\{\mathcal{R}, (\lambda - \frac{1}{\lambda})(t^3 \otimes t^2) - (\mu - \frac{1}{\mu})(t^2 \otimes t^3)\right\} = 0
\end{aligned} \tag{23}$$

The $\pm$ in the last equation are for the trigonometric and hyperbolic cases respectively. We will now invoke the reality condition of the Lax operator (a consequence of time reversal invariance) which translates in this case to the hermiticity of the generators t^i. This is guaranteed if we take the Lie algebra to be compact. Braiding non-triviality also requires that it should be semi-simple. The only situation with three non-commuting generators is the $su(2)$ algebra.

To check the consistency of these equations, we first notice that for $\beta^2 = 0$ one has a classical situation, i.e. $\mathcal{R} = 1$. If the classical limit is smooth, then we can have a perturbative expansion of $\mathcal{R} = 1 + i\beta^2\mathcal{R}_1 + \cdots$ and then $\mathcal{R}_1$ would satisfy the same relations as above except that the anticommutators in the right hand side would be replaced by $-\beta^2[(\lambda+1/\lambda)(t^1\otimes t^2) - (\mu+1/\mu)(t^2\otimes t^1)]$ and $\pm\beta^2[(\lambda-1/\lambda)(t^3\otimes t^2) - (\mu-1/\mu)(t^2\otimes t^3)]$. The most general form of $\mathcal{R}_1$ consistent with the vanishing commutator (the first of Eq.(23) is

$$\mathcal{R}_1 = a(t^2\otimes t^2) + b(t^3\otimes t^3 + t^1\otimes t^1)$$

To determine the unknown coefficients a and b one just substitutes this in the next equation. The linear independence of $t^1\otimes t^2$ and $t^2\otimes t^1$ would give rise to two linear equations from which a and b can be obtained uniquely. This can surely be not consistent with both forms of the third equation (one for trigonometric and the other for the hyperbolic)! A few steps of simple algebra shows that only the trigonometric case is consistent.

If one expands the trace of the scattering transform independently around $\lambda = 0$ and $\lambda^{-1} = 0$ (this is definitely possible in the classical limit where one can use the usual analytic continuation [4]) the co-efficient of λ and λ^{-1} added together would transform like the time component of momentum vector and hence it can be identified with the Hamiltonian. It exactly coincides with the Hamiltonian of a sine-Gordon field.

One thus concludes that for a single pseudo-scalar field (and its canonical conjugate) the only consistent quantum field theory permitting a nontrivial braiding of the scattering transforms is the one whose Lax operator is of the form of Eq.(21). This is our first result.

The exact solution of Eq.(23) will be representation dependent as anti-commutation relation among the generators or their direct products are not governed by Lie algebra alone. However the exact correspondence of the degrees of freedom between the original theory (local fields ϕ and $\dot{\phi}$) and the transformed theory (the off-diagonal elements of the scattering transform) can be invoked only when one considers a 2×2 representation of $\mathcal{T}$ since there will then be only two off-diagonal elements. In this representation (fundamental 2×2 representation) the master equations can be solved immediately since different Pauli matrices anti-commute and the solution is

$$\begin{aligned}\mathcal{R}(\lambda,\mu) \quad = \quad & (1\otimes 1) - i\beta^2\left[\frac{\frac{\lambda}{\mu}(1-i\frac{\beta^2}{4}) + \frac{\mu}{\lambda}(1+i\frac{\beta^2}{4})}{\frac{\lambda}{\mu}(1-i\frac{\beta^2}{4}) - \frac{\mu}{\lambda}(1+i\frac{\beta^2}{4})}(t^2\otimes t^2)\right. \\ & \left. + \frac{2}{\frac{\lambda}{\mu}(1-i\frac{\beta^2}{4}) - \frac{\mu}{\lambda}(1+i\frac{\beta^2}{4})}\left\{(t^3\otimes t^3) + (t^1\otimes t^1)\right\}\right]\end{aligned} \tag{24}$$

This braiding matrix for the sine-Gordon model has been known for a long time [7, 8] and we just demonstrated here how naturally it occurs in relativistic field theories of pseudo-scalars.

Extension to multi-component pseudo-scalars can be performed by associating sine, cosine functions and the time derivatives of each of the pseudo-scalar with the $su(2)$ sub-algebras of the root systems of an appropriate large Lie algebra. However consistent solutions are known to exist only when the roots are either simple or the one associated with the lowest height [9]. The Hamiltonian correspond to the Toda model which is basically of sine-Gordon type.

'Spin' 1/2 Dirac Fields:

To start with we will assume that all the fields commute rather than anti-commute for space like separation. This will ensure that without the singularities the transposition of the scattering transforms are trivial. Even if there are anticommuting fields, in one spatial dimension where a natural ordering exists with the value of the coordinate, a simple Jordan Wigner transformation

$$\psi \rightarrow \left[e^{\pm i\pi \int_{-\infty}^{x^1} j_0(\xi) d\xi} \right] \psi \tag{25}$$

switches a commutator to anti-commutator and vice versa among two complex fields with space like separation.

The relevant OPE algebras are

$$\psi_i(x^1, x^0)\psi_j^\dagger(y^1, x^0) = \frac{(-1)^{j+1}\delta_{ij}}{2\pi i(x^1 - y^1 - i(-1)^j 0^+)} + \text{ less singular terms} \tag{26}$$

where i, j are the Dirac spinor indices. This is consistent with the irreducible LT of the Dirac spinor components given in Eq.(7) and the equal time commutation algebra

$$[\psi_i(x^1, x^0), \psi_j^\dagger(y^1, x^0)] = \delta_{ij}\delta(x^1 - y^1)$$

A Lorentz covariant (including parity and time reversal) choice of $L(x, \lambda)$ could be

$$L(x, \lambda) = \left[\frac{m}{2}\left(\lambda - \frac{1}{\lambda}\right) + aj_1 \right] t^3 + b\left(\sqrt{\lambda}\psi_1 - \sqrt{\frac{1}{\lambda}}\psi_2\right) t_+ + b^*\left(\sqrt{\lambda}\psi_1^\dagger - \sqrt{\frac{1}{\lambda}}\psi_2^\dagger\right) t_- \tag{27}$$

where $j_1 =: \psi_1^\dagger\psi_1 : - : \psi_2^\dagger\psi_2 :$ is the space component of the current vector. The two derived OPE relevant for the subsequent study are

$$T(j_1(x)\psi_k(y)) = (-1)^k \frac{\psi_k(y)}{2\pi i(x - y - i0^+)} + \cdots \tag{28}$$

It should be noted that the OPE of $j_1(x)$ with $j_1(y)$ does not produce any singularity in the equal time limit (even though $j_0(x)$ with $j_1(y)$ would have a Schwinger term about which we will discuss later). While constructing L We could have included higher tensors but that would not be consistent with the closure property. It should further be noted that we have included only those fields for which the OPE singularities are no more than simple poles. As a consequence we can proceed, as before, to obtain the set of master equations for the

braiding matrix for the scattering transform by comparing the product of infinitesimal strings $[\mathcal{T}_\epsilon(x;\lambda)\otimes 1][1\otimes\mathcal{T}_\epsilon(x;\mu)]$ with the product in opposite order after taking into consideration all quantum corrections arising out of singularities of the OPE given in Eq.(26) and (28). The quantum correction terms will only have identity, $\psi_{1,2}$ and $\psi^\dagger_{1,2}$ field operators. The three independent master equations (others essentially describing the complex conjugation and some discrete symmetry nature of the braiding matrix) are

$$\begin{aligned}
&[\mathcal{R},(t^3\otimes 1)+(1\otimes t^3)]=0\\
&\left[\mathcal{R},\left(\lambda-\frac{1}{\lambda}\right)(t^3\otimes 1)\quad+\left(\mu-\frac{1}{\mu}\right)(1\otimes t^3)\right]\\
&\qquad\qquad +i|b|^2\left(\sqrt{\lambda\mu}+\frac{1}{\sqrt{\lambda\mu}}\right)\{\mathcal{R},(t_-\otimes t_+)-(t_+\otimes t_-)\}=0\\
&\left[\mathcal{R},\sqrt{\lambda}(t_-\otimes 1)+\sqrt{\mu}(1\otimes t_-)\right]-\frac{a}{2}\left\{\mathcal{R},\sqrt{\mu}(t^3\otimes t_-)-\sqrt{\lambda}(t_-\otimes t^3)\right\}=0
\end{aligned}\tag{29}$$

With the obvious form (consistent with the first equation of Eq.(29)

$$\mathcal{R}(\lambda,\mu)=1\otimes 1+X(\lambda,\mu)t_3\otimes t_3+Y(\lambda,\mu)\left[t_+\otimes t_-+t_-\otimes t_+\right]$$

we can solve for the co-efficients $X(\lambda,\mu)$ and $Y(\lambda,\mu)$ from the remaining equations of Eq.(29). It turns out, by using the linear independence of different $t_i\otimes t_j$, that there are one too many relations and the consistency can be restored provided

$$|b|^2=m\left(\frac{a}{1+a^2/16}\right)\tag{30}$$

The exact expression for the braiding matrix turns out to be

$$\begin{aligned}
\mathcal{R}(\lambda,\mu)=1\otimes 1\quad &-ia\left[\frac{\sqrt{\frac{\lambda}{\mu}}\left(1-\frac{ia}{4}\right)+\sqrt{\frac{\mu}{\lambda}}\left(1+\frac{ia}{4}\right)}{\sqrt{\frac{\lambda}{\mu}}\left(1-\frac{ia}{4}\right)-\sqrt{\frac{\mu}{\lambda}}\left(1+\frac{ia}{4}\right)}(t_3\otimes t_3)\right.\\
&\left.+\frac{2}{\sqrt{\frac{\lambda}{\mu}}\left(1-\frac{ia}{4}\right)-\sqrt{\frac{\mu}{\lambda}}\left(1+\frac{ia}{4}\right)}\{(t_1\otimes t_1)+(t_2\otimes t_2)\}\right]
\end{aligned}\tag{31}$$

The existence of the braiding matrix implies the existence of infinite number of operators in involution. They are the co-efficients of the expansion around $\lambda=0$ and ∞ around which the diagonal elements of $\mathcal{T}(\lambda)$ is analytic. The linear combination of the co-efficient of λ and λ_1 would transform, under LT, like the Hamiltonian. A classical evaluation of this combination yields the Hamiltonian of the massive Thirring model. One might think at this stage that the model is soluble since the form of the braiding matrix is exactly known [10]. But there is a serious problem. With 2×2 matrix representation of $\mathcal{T}$ one would land up with only two scattering transform variables whereas the independent local fields are four in number (ψ_1,ψ_1 and their complex conjugates). We will address this problem in the next section.

Chiral Extension:

If one relaxes the parity invariance constraint of the scattering transform and replaces it by parity covariance, one gets the interesting possibility of including true scalar fields in this framework. In this case one will have two such transforms, one the parity image of the other, each being invariant under proper Lorentz transformations. One can now include both $\partial_x\phi$ and $\dot{\phi}$ as vector valued fields in L. For a true scalar field ϕ, $\partial_x\phi$ is the space component of a true vector while $\dot{\phi}$ is the space component of an axial vector $\epsilon^{\mu\nu}\partial_\nu\phi$. For the pseudoscalar field it would just be the opposite. The simplest form of L consistent with the closure of OPE is of the form

$$L_+(\xi,\lambda) = m\lambda t_3 + (\alpha\partial_x\phi + \beta\dot{\phi})t_2 \tag{32}$$

and its parity counterpart

$$L_-(\xi,\lambda) = \frac{m}{\lambda}t_3 + (-\alpha\partial_x\phi + \beta\dot{\phi})t_2 \tag{33}$$

The immediate difficulty one encounters here is that the OPE of $\dot{\phi}$ and $\partial_x\phi$ now contains a second order pole (Schwinger term) and the method prescribed earlier to evaluate the quantum correction for the direct product of two strings would not work. In literature this is called non-ultralocality. A simple way to get around this problem is to identify the $\mathcal{T}$'s as gauge transformation of another set of $\tilde{\mathcal{T}}$'s whose OPE's will not involve such Schwinger terms. Such a possibility was invoked for discrete systems some time back [11].

For example the $\mathcal{T}(x,y,\lambda)$ associated with Eq.(32) can be written as

$$\mathcal{T}_+(x,y,\lambda) = e^{i\alpha\phi(x)t_2}\tilde{\mathcal{T}}_+(x,y.\lambda)e^{-i\alpha\phi(y)t_2} \tag{34}$$

with

$$\begin{aligned}\tilde{\mathcal{T}}_+(x,y.\lambda) &= \mathcal{P}e^{i\int_y^x \tilde{L}_+(\xi,\lambda)d\xi}\\ \tilde{L}_+(\xi,\lambda) &= \beta\dot{\phi}(\xi)t_2 + \lambda\cos(\alpha\phi(\xi))t_3 + \lambda\sin(\alpha\phi(\xi))t_1\end{aligned} \tag{35}$$

The exponential factors at the two edges of the string are to be understood as space-like separated from the fields inside $\tilde{\mathcal{T}}$ and hence the whole object is manifestly normal ordered. The local operator $\tilde{L}_+$ looks very similar to what we have already discussed earlier and clearly the OPE's among different $\tilde{L}_+$'s do not involve singularities above simple poles. The quantum correction terms from their OPE's would therefore be similar to what we have obtained before. There will now be additional correction terms coming from the OPE of $\dot{\phi}$ in $\tilde{L}_+$'s with the exponential terms at the edges of the other strings. To evaluate the contribution to the quantum corrections coming from the edge terms we follow the convention that in the OPE of strings the one occurring at the right is infinitesimally shifted down compared to the one preceding it. Thus we can proceed as before with product of two infinitesimal strings of length ϵ

$$\begin{aligned}&(\mathcal{T}_+(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2},\lambda)\otimes 1)(1\otimes\mathcal{T}_+(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2},\mu)\\ &= :e^{i\alpha\phi(x+\frac{\epsilon}{2})(t_2\otimes 1)}\left(1\otimes 1 + i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}}[\tilde{L}_+(\xi,\lambda)\otimes 1]d\xi\right)e^{-i\alpha\phi(x-\frac{\epsilon}{2})(t_2\otimes 1)}:\\ &:e^{i\alpha\phi(x_-+\frac{\epsilon}{2})(1\otimes t_2)}\left(1\otimes 1 + i\int_{x_--\frac{\epsilon}{2}}^{x_-+\frac{\epsilon}{2}}[1\otimes\tilde{L}_+(\eta,\mu)]d\eta\right)e^{-i\alpha\phi(x_--\frac{\epsilon}{2})(1\otimes t_2)}:\end{aligned} \tag{36}$$

and use the OPE

$$e^{iK\phi(\xi)}\dot{\phi}(\eta) = \frac{e^{iK\phi(\xi)}}{\xi-\eta-i\pi 0^+} + :e^{iK\phi(\xi)}\dot{\phi}(\eta):$$

Because of the shifting convention the edges $x+\frac{\epsilon}{2}$ and $x_- - \frac{\epsilon}{2}$ are always space-like separated from all the fields occurring within the two strings and hence will not have any contribution to any quantum correction. Therefore the only edge contributions come from (a) the product of $exp(-i\alpha\phi(x-\frac{\epsilon}{2})(t_2\otimes 1))$ with $\beta\dot{\phi}(\eta)(1\otimes t_2)$ and (b) the product of $\beta\dot{\phi}(\xi)(t_2\otimes 1)$ with $exp(i\alpha\phi(x_- + \frac{\epsilon}{2})(1\otimes t_2))$.

Apart from these edge contributions there are the usual quantum corrections coming from the OPE's of $\dot{\phi}$'s with $\sin(\alpha\phi)$'s and $\cos(\alpha\phi)$'s as discussed earlier. The final result to order ϵ is

$$
\begin{aligned}
&\left(\mathcal{T}_+(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2},\lambda)\otimes 1\right)\left(1\otimes\mathcal{T}_+(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2},\mu)\right)\\
&=: e^{i\alpha\phi(x+\frac{\epsilon}{2})[(t_2\otimes 1)+(1\otimes t_2)]}\left[\left(1\otimes 1+\frac{i\alpha\beta}{2}(t_2\otimes t_2)\right)^2\right.\\
&+i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[\tilde{L}_+(\xi,\lambda)\otimes 1]\left(1\otimes 1+\frac{i\alpha\beta}{2}(t_2\otimes t_2)\right)\\
&+i\left(1\otimes 1+\frac{i\alpha\beta}{2}(t_2\otimes t_2)\right)\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[1\otimes\tilde{L}_+(\xi,\mu)]\\
&-\frac{i\alpha\beta}{2}\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[\cos\alpha\phi(\xi)(\lambda(t_1\otimes t_2)-\mu(t_2\otimes t_1))]\\
&\left.+\frac{i\alpha\beta}{2}\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[\sin\alpha\phi(\xi)(\lambda(t_3\otimes t_2)-\mu(t_2\otimes t_3))]\right] e^{-i\alpha\phi(x-\frac{\epsilon}{2})[(t_2\otimes 1)+(1\otimes t_2)]}:
\end{aligned}
\tag{37}
$$

The product in reverse order can be evaluated in a similar fashion to order ϵ (again keeping in mind the convention of shifting the edges).

$$
\begin{aligned}
&\left(1\otimes\mathcal{T}_+(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2},\mu)\right)\left(\mathcal{T}_+(x+\tfrac{\epsilon}{2},x-\tfrac{\epsilon}{2},\lambda)\otimes 1\right)\\
&=: e^{i\alpha\phi(x+\frac{\epsilon}{2})[(t_2\otimes 1)+(1\otimes t_2)]}\left[\left(1\otimes 1+\frac{i\alpha\beta}{2}(t_2\otimes t_2)\right)^2\right.\\
&+i\left(1\otimes 1+\frac{i\alpha\beta}{2}(t_2\otimes t_2)\right)\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[\tilde{L}_+(\xi,\lambda)\otimes 1]\\
&+i\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[1\otimes\tilde{L}_+(\xi,\mu)]\left(1\otimes 1+\frac{i\alpha\beta}{2}(t_2\otimes t_2)\right)\\
&+\frac{i\alpha\beta}{2}\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[\cos\alpha\phi(\xi)(\lambda(t_1\otimes t_2)-\mu(t_2\otimes t_1))]\\
&\left.-\frac{i\alpha\beta}{2}\int_{x-\frac{\epsilon}{2}}^{x+\frac{\epsilon}{2}} d\xi[\sin\alpha\phi(\xi)(\lambda(t_3\otimes t_2)-\mu(t_2\otimes t_3))]\right] e^{-i\alpha\phi(x-\frac{\epsilon}{2})[(t_2\otimes 1)+(1\otimes t_2)]}:
\end{aligned}
\tag{38}
$$

To find the braiding relation, if any, one has to solve for a $\mathcal{R}$ matrix that would connect Eq.(37) with Eq.(38). This would, in turn, by virtue of the linear independence of the fields, lead to a set of algebraic master equations for R.

As before, comparison of terms proportional to $\dot{\phi}$ gives

$$[\mathcal{R}(\lambda,\mu),(t_2\otimes 1)+(1\otimes t_2)]=0 \tag{39}$$

and comparison of terms proportional to $\cos(\alpha\phi)$ gives

$$[\mathcal{R}(\lambda,\mu),\lambda(t_3\otimes 1)-\mu(1\otimes t_3)]=\frac{\alpha\beta}{4}\left\{\mathcal{R},\lambda(t_1\otimes t_2)-\mu(t_2\otimes t_1)\right\} \tag{40}$$

Comparison of terms proportional to $\sin(\alpha\phi)$ gives no new relation (only reflecting the automorphism symmetry of the t's).

The master equations are very similar to the one we obtained for the sine-Gordon theory (Eq.(23. The coupling constant is $\alpha\beta/4$ instead of $\beta^2/2$ and the spectral parameters are directly λ and μ instead of momentum and energy. The resultant $R(\lambda,\mu)$ however has identical form

$$\begin{aligned}
\mathcal{R}(\lambda,\mu) &= 1\otimes 1 + A(\lambda,\mu)t_2\otimes t_2 + B(\lambda,\mu)(t_3\otimes t_3 + t_1\otimes t_1)\\
A(\lambda,\mu) &= -\frac{i\alpha\beta}{2}\frac{\frac{\lambda}{\mu}\left(1-\frac{i\alpha\beta}{8}\right)+\frac{\mu}{\lambda}\left(1+\frac{i\alpha\beta}{8}\right)}{\frac{\lambda}{\mu}\left(1-\frac{i\alpha\beta}{8}\right)-\frac{\mu}{\lambda}\left(1+\frac{i\alpha\beta}{8}\right)}\\
B(\lambda,\mu) &= -\frac{i\alpha\beta}{2}\frac{2}{\frac{\lambda}{\mu}\left(1-\frac{i\alpha\beta}{8}\right)-\frac{\mu}{\lambda}\left(1+\frac{i\alpha\beta}{8}\right)}
\end{aligned} \tag{41}$$

That the scattering transform corresponding to Eq.(32) would lead to a classical integrable system similar to the sine-Gordon theory was well-known [4] and therefore it is not surprising that the braiding matrix for quantum theory of them would be of the same form. The only difference is that the parity conservation would make it necessary to use the scattering transforms corresponding to both Eq.(32) and Eq.(33) to describe one single system. It is therefore a chiral theory having twice the degrees of freedom compared to the ordinary sine-Gordon theory. That it is really a chiral theory can be checked by the following three steps.

(a) The scattering transforms T_+ corresponding to Eq.(32) have already been shown to satisfy the braiding relation

$$\mathcal{R}_+(\lambda,\mu)\,[\mathcal{T}_+(x,y;\lambda)\otimes 1]\,[1\otimes\mathcal{T}_+(x,y;\mu)] = [1\otimes\mathcal{T}_+(x,y;\mu)]\,[\mathcal{T}_+(x,y;\lambda)\otimes 1]\,\mathcal{R}_+(\lambda,\mu) \tag{42}$$

with $\mathcal{R}_+(\lambda,\mu)$ given by Eq.(41).

(b) In an identical way one can obtain the braiding relations among the scattering transforms $\mathcal{T}_-$ corresponding to Eq.(33) by noting that this too is a gauge transform of an ultra-local theory

$$\mathcal{T}_-(x,y,\lambda) = e^{-i\alpha\phi(x)t_2}\tilde{\mathcal{T}}_-(x,y.\lambda)e^{i\alpha\phi(y)t_2} \tag{43}$$

with

$$\begin{aligned}
\tilde{\mathcal{T}}_-(x,y.\lambda) &= \mathcal{P}e^{i\int_y^x \tilde{L}_-(\xi,\lambda)d\xi}\\
\tilde{L}_-(\xi,\lambda) &= \beta\dot{\phi}(\xi)t_2 + \lambda\cos(\alpha\phi(\xi))t_3 - \lambda\sin(\alpha\phi(\xi))t_1
\end{aligned} \tag{44}$$

This would lead to

$$\mathcal{R}_-(\lambda,\mu)\,[\mathcal{T}_-(x,y;\lambda)\otimes 1]\,[1\otimes\mathcal{T}_-(x,y;\mu)] = [1\otimes\mathcal{T}_-(x,y;\mu)]\,[\mathcal{T}_-(x,y;\lambda)\otimes 1]\,\mathcal{R}_-(\lambda,\mu) \tag{45}$$

with $\mathcal{R}_-(\lambda,\mu)$ having the similar form as of $\mathcal{R}_+(\lambda,\mu)$ except $\alpha\beta$ is to be replaced by $-\alpha\beta$ and λ interchanged with μ.

(c) The mixed braiding among $\mathcal{T}_+$ and $\mathcal{T}_-$ turn out to be trivial in the sense that the braiding matrix does not depend on the spectral parameters. This is an unexpected result and needs a little elaboration. Notice the differences in signs before the $\sin(\alpha\phi)$ in the Eq.(35) and Eq.(44). It is this feature that is reflected in the quantum correction in the

mixed product in a form slightly different from what we have got in Eq.(37) and in Eq.(38). If one uses the relations (true only in the fundamental representation of the t's)

$$(t_2 \otimes t_3) = \frac{-i}{2}(1 \otimes t_1)(t_2 \otimes t_2),$$

$$(t_2 \otimes t_1) = \frac{i}{2}(1 \otimes t_3)(t_2 \otimes t_2),$$

and the fact that $(t_2 \otimes t_2)$ anticommutes with $(1 \otimes t_{1,3})$ as well as $(t_{1,3} \otimes 1)$, we will get the master equations for the $\mathcal{R}_{\mathrm{mixed}}$ by comparing separately the co-efficients of the fields as

$$\begin{aligned}
&[\mathcal{R}_{\mathrm{mixed}}, (t_2 \otimes 1) \pm (1 \otimes t_2)] = 0, \\
&[i\mathcal{R}_{\mathrm{mixed}}, \lambda(t_3 \otimes 1) + \mu(1 \otimes t_3)] \\
&+ \left[\tfrac{\alpha\beta}{2}(t_2 \otimes t_2)R_{\mathrm{mixed}}, \lambda(t_3 \otimes 1) + \mu(1 \otimes t_3)\right] = 0, \\
&[i\mathcal{R}_{\mathrm{mixed}}, \lambda(t_1 \otimes 1) - \mu(1 \otimes t_1)] \\
&+ \left[\tfrac{\alpha\beta}{2}(t_2 \otimes t_2)\mathcal{R}_{\mathrm{mixed}}, \lambda(t_1 \otimes 1) - \mu(1 \otimes t_3)\right] = 0
\end{aligned} \tag{46}$$

These equations tell that $(1 - \frac{i\alpha\beta}{2}(t_2 \otimes t_2))R_{\mathrm{mixed}}$ must be proportional to identity. Consequently, since overall normalisation of R is immaterial, we get

$$\mathcal{R}_{\mathrm{mixed}} = 1 + \frac{i\alpha\beta}{2}(t_2 \otimes t_2) \tag{47}$$

The triviality of $\mathcal{R}_{\mathrm{mixed}}$ means that the all the elements of the scattering transforms of one chirality essentially commute with all the elements of the other chirality. This is what is really expected out of a two-component chiral theory. The fact that the action variables (the diagonal elements of $\mathcal{T}$ - the constants of motion) of one chirality commutes with the angle variables (the non-diagonal elements - the creation and destruction operators) of the other chirality implies that the chiral states do not mix during time evolution. In spite of a mass scale present in the theory the model still exhibits a characteristic of a massless theory, namely, the conservation of chirality (γ^5 invariance).

A similar chiral extension is also possible for a Dirac Theory of Eq.(27) where gj_1 can be replaced by $gj_1 \pm \alpha j_0$. A very general expression for a chiral L could be

$$L(x,\lambda) = \left(k_1 + \gamma^5\beta k_0 + aj_1 + \gamma^5\alpha j_0\right) t_3 + \beta' \left(\sqrt{\lambda}\psi_1 - \beta''\gamma^5\sqrt{\frac{1}{\lambda}}\psi_2\right) t_+ + \mathrm{h.c.} \tag{48}$$

Once again one can gauge relate such a theory with the integrable system which we have already discussed in the parity invariant Dirac case. For a consistent solution for the braiding matrix it turns out that the new parameters $\beta,\ \beta',\ \beta''$ and α are related. Now one will have a two component chiral description for both the creation and destruction parts and their algebra with the two-fold sets of constants of motion. If such a problem can be quantised consistently one would have a true solution for a Massive Thirring model. In contrast to the case of scalar chiral fields this time, however, it would not describe a decoupled theory. The detailed dynamical studies of such two-fold chiral integrable systems would be carried out in a latter article.

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